MATH4060 Solution 1

February 2023

Exercise 1

(a) The assumption $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$ implies that

$$A(\xi) - B(\xi) = e^{2\pi i \xi t} \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx = 0.$$

(b) Consider A defined on the upper half plane. Note that for $z = u + iv$, $v > 0$, and $x \leq t$, we have

$$|f(x) e^{-2\pi i z(x-t)}| = |f(x)| e^{2\pi v(x-t)} \leq |f(x)|.$$

By the moderate decrease of f, $A(z)$ is well-defined and bounded. To see that A is holomorphic on the upper half plane, we argue as in Theorem 3.1: define $A_n(z) = \int_{-n}^{t} f(x) e^{-2\pi i z(x-t)} \, dx$ and observe that $A_n \to A$ uniformly because $|A_n(z) - A(z)| \leq \int_{-\infty}^{-n} |f(x)| \, dx$ and f has moderate decrease. Each A_n is holomorphic by Theorem 5.4 of Chapter 2 and so is the uniform limit A.

Similarly, B is holomorphic and bounded on the lower half plane. Part (a) and the symmetry principle (Theorem 5.5 of Chapter 2) imply that F is entire and bounded, hence a constant. In fact this constant is 0, since the boundedness of f implies that (on the upper half plane)

$$|A(u + iv)| \leq C \int_{-\infty}^{t} e^{2\pi v(x-t)} \, dx = \frac{C}{2\pi v},$$

so that $A(z) \to 0$ as $\operatorname{Im}(z) \to \infty$.

(c) The first statement follows from $F(0) = 0$ and the second from the continuity of f.

Exercise 3

Consider the function $f(z) = \frac{a}{a^2+z^2} e^{-2\pi i z \xi}$ having simple poles at $z = \pm ai$ with residue $\pm(2i)^{-1} e^{\pm 2\pi a \xi}$. When $\xi \geq 0$, consider the contour from $-R$ to R along the real axis and then from R to $-R$ along the semicircular arc C_R^- in the lower half plane. Along the arc $z = Re^{i\theta}$ (with $\operatorname{Im}(z) < 0$ and assume $R > a$),

$$|f(z)| = \frac{a}{|a^2 + z^2|} e^{2\pi \operatorname{Im}(z) \xi} \leq \frac{a}{R^2 - a^2}.$$

So $\int_{C_R^-} f(z) \, dz \to 0$ as $R \to \infty$. Because the contour is clockwise oriented, the residue theorem implies that

$$\int_{-\infty}^{\infty} \frac{a}{a^2 + x^2} e^{-2\pi i x \xi} \, dx = -2\pi i \operatorname{res}_{z=-ai} f = \pi e^{-2\pi a \xi} = \pi e^{-2\pi a |\xi|}.$$

The case $\xi < 0$ is similar and uses the semicircular contour on the upper half plane. The second statement of the question is by direct integration.

Exercise 7

(a) We first compute $\hat{f}(\xi)$ using residue theorem. Consider the function $g(z) = (\tau + z)^{-k} e^{-2\pi i \xi z}$, with an order k pole at $z = -\tau$ (in the lower half plane) with

$$\operatorname{res}_{z=-\tau} g = \frac{1}{(k-1)!}\left(\frac{d}{dz}\right)^{k-1} e^{-2\pi i \xi z}\Big|_{z=-\tau} = \frac{(-2\pi i \xi)^{k-1}}{(k-1)!} e^{2\pi i \xi \tau}.$$

Similar to exercise 3, for $\xi > 0$, consider the semicircular contour in the lower half plane. Since $k \geq 2$, the same argument shows that $\int_{C_R^-} g(z)\, dz \to 0$ as $R \to \infty$, and thus the residue theorem gives

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i \xi x}}{(\tau + x)^k}\, dx = -2\pi i \operatorname{res}_{z=-\tau} g = \frac{(-2\pi i)^k}{(k-1)!} \xi^{k-1} e^{2\pi i \xi \tau}.$$

For $\xi \leq 0$, the same argument in the upper half plane shows that $\hat{f}(\xi) = 0$ because the contour does not enclose the pole. The desired identity is now a direct consequence of the Poisson summation formula.

(b) Apply (a) with $k = 2$. Note that $|e^{2\pi i \tau}| < 1$ since $\operatorname{Im} \tau > 0$, so we have the following identity (viewing as a function of τ):

$$\begin{aligned}\sum_{m=1}^{\infty} m e^{2\pi i m \tau} &= \left(\frac{1}{2\pi i} \sum_{m=0}^{\infty} e^{2\pi i m \tau}\right)' = \left(\frac{1}{2\pi i (1 - e^{2\pi i \tau})}\right)' \\ &= \frac{e^{2\pi i \tau}}{(1 - e^{2\pi i \tau})^2} = \frac{1}{(e^{-\pi i \tau} - e^{\pi i \tau})^2} = -\frac{1}{4 \sin^2(\pi \tau)}.\end{aligned}$$

(c) Yes, because both sides are meromorphic functions on $\mathbb{C}$ that have the same poles and agree on the upper half plane.

Exercise 10

Let $l > 0$. First note that for $z = x + it$ and $\zeta = \xi + i\eta \in S_l$ (i.e. $|\eta| < l$), we have

$$|f(z) e^{-2\pi i z \zeta}| = |f(x+it)| e^{2\pi (x\eta + t\xi)} \leq c e^{-ax^2 + 2\pi x \eta} e^{bt^2 + 2\pi t \xi} \tag{1a}$$

$$\leq c e^{-ax^2 + 2\pi l |x|} e^{bt^2 + 2\pi t \xi} \leq c_1 e^{-\tilde{a} x^2} e^{bt^2 + 2\pi t \xi}, \tag{1b}$$

for any $0 < \tilde{a} < a$ and some constant c_1 independent of x and $\zeta \in S_l$ (but dependent on $a, \tilde{a}, l$). Similar to Theorem 3.1, observe that $\hat{f}(\zeta)$ is holomorphic in every S_l: let $\hat{f}_n(\zeta) = \int_{-n}^{n} f(x) e^{-2\pi i x \zeta}\, dx$, each holomorphic by Theorem 5.4 of Chapter 2. Equation (1b) with $t = 0$ and the integrability of $e^{-\tilde{a} x^2}$ imply that $|\hat{f}_n(\zeta) - \hat{f}(\zeta)| \leq c_1 \int_{|x| \geq n} e^{-\tilde{a} x^2}\, dx \to 0$ uniformly in $\zeta \in S_l$, as $n \to \infty$. So $\hat{f}$ is holomorphic by Theorem 5.2 of Chapter 2.

Next, with ζ fixed, we show that the contour of integration can be changed to $\{\operatorname{Im}(z) = y\}$ for any fixed $y \in \mathbb{R}$, i.e. we have

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \zeta}\, dx = \int_{-\infty}^{\infty} f(x + iy) e^{-2\pi i (x + iy) \zeta}\, dx. \tag{2}$$

(The integrals on both sides are well-defined by (1b).) To prove (2) when $y \neq 0$, consider the entire function $f(z) e^{-2\pi z \zeta}$ and the rectangular contour defined by the vertices $-R, R, R + iy, -R + iy$. As in the proof[1] of Theorem 2.1, it suffices to show that the integrals along the vertical segments of the contour tends to 0 as $R \to \infty$. By (1b), along the left vertical segment, we have

$$\left| \int_0^y f(-R + it) e^{-2\pi i (-R + it) \zeta}\, dt \right| \leq C e^{-\tilde{a} R^2} \to 0$$

[1] In fact, if $\eta = 0$, we could just apply the proof in Theorem 2.1.

as $R \to \infty$. And the same holds for the right vertical segment. This proves (2).

Finally, we estimate $|\hat{f}(\zeta)|$ using the shifted contour: by (1a), we have

$$\begin{aligned} |\hat{f}(\zeta)| &\leq ce^{by^2+2\pi y\xi} \int_{-\infty}^{\infty} e^{-ax^2+2\pi x\eta}\, dx \\ &= ce^{by^2+2\pi y\xi} e^{b'\eta^2} \int_{-\infty}^{\infty} e^{-a(x-\sqrt{b'/a}\eta)^2}\, dx \\ &\leq c' e^{by^2+2\pi y\xi} e^{b'\eta^2}, \end{aligned}$$

where $b' > 0$ is obtained by completing square (and is independent of η); while c' is some constant also independent of $\zeta = \xi + i\eta$. Consider a sufficiently small $d > 0$ so that $a' := 2\pi d - bd^2 > 0$, and then take $y = -d\xi$ in the above to obtain the desired estimate

$$|\hat{f}(\zeta)| \leq c' e^{-a'\xi^2+b'\eta^2}.$$